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## Optimizing racing wheelchair design through coupled biomechanical-mechanical simulation

Adrien Loiseau<sup>1</sup>, Thibault Marsan<sup>2[0000-0002-5582-0271]</sup>, Pablo Navarro<sup>1[0000-0002-3351-0109]</sup>, Bruno Watier<sup>2[0000-0003-3146-2884]</sup>, Yann Landon<sup>1[0000-0002-3822-2421]</sup>,

<sup>1</sup> Institut Clément Ader (ICA), Université de Toulouse, CNRS, IMT Mines Albi, INSA, ISAE-SUPAERO, UPS, 3 rue Caroline Aigle, 31400 Toulouse, France
 <sup>2</sup> LAAS-CNRS, Université de Toulouse, CNRS, UPS, Toulouse, France

**Abstract.** The purpose of this study is to optimize the design of racing wheel-chairs to improve the performances of the athletes. The design of manual wheel-chair allows athletes to express their full potential. Two models have then been created. The first one to compute the optimal position of the shoulder of the athlete relatively to the wheelchair to obtain the maximal wheelchair speed for long distance races. The second one was designed to represent the 100 m race and to optimize the pelvis position of the athlete on the wheelchair to reduce the time to reach 100 m. Our model quantified the maximal speed reached by the wheelchair to 32 km/h and the optimal time to 14.35 s. To obtain these performances, the athlete would be in a lying position, with the vertical position of the pelvis centre close to the vertical position of the shoulder. The second program also returned the optimal speed curve of the wheelchair during the 100 m race. The coaches could then use the optimal acceleration curve found in this study to match the acceleration of the wheelchair of their athlete.

**Keywords:** Wheelchair design; Racing; Athlete positioning; Para athletics

#### 1 Introduction

Manual wheelchair is a key element in the performance of wheelchair racing athletes. For this reason, the design of such wheelchairs is a particular issue for high performance athletes. Many advances have been made since the first racing wheelchair [1] but improvements remained to be made. The purpose of this study was to optimize the design of racing wheelchairs to improve the performance of the athlete.

As a design requirement, the optimal position of the athlete in his wheelchair must be determined for each sport and each athlete, as well as the global mass repartition [2]. Indeed, in wheelchair racing the athletes present very different pathologies which necessitate customized settings. The work presented in this paper concerns athletics, especially wheelchair racing. The manual wheelchair designed for racing is specific to this sport. At first, it was just a modified wheelchair [3]. One of the major changes in the manual racing wheelchair was the evolution of a four-wheeled wheelchair to a three-wheeled wheelchair [4], reducing the total weight of the wheelchair. Then, the materials evolved [5], reducing the weight once more to make the wheelchair easier to propel and

maneuver. It would then be interesting to study the optimal position of the athlete on the wheelchair, that would allow him to express his full potential.

It has already been demonstrated that the position on the wheelchair would influence both the pushing pattern and the kinematics of the subject [6]. Different methods have been used in the literature. Experimental data were collected using inertia rollers, inertial measurement units, or motion capture [7]. Propulsion models have also been developed [8]. It has been stated that, since most of the motion occurred in the sagittal plane, a 2D model of the wheelchair would be sufficient to study the wheelchairs' propulsion. Most of the developed models in the literature were used to compute optimal parameters of the wheelchair. However, none of the models of the literature have been used to obtain the optimal position of the athlete on racing wheelchair.

Para-athletes compete in different racing length, from 100 m to marathons, and the athletes' kinematics changes depending on the race. For instance, T53 athletes, which is a category of para-athletes with abdominal deficiencies who have little to no control over their trunk and their legs, use their shoulder to propel the wheelchair during the first acceleration but not once they reached their maximal speed [9]. However, T54 athletes, who also have low control over their legs but can control their trunk, use their shoulders both during acceleration and while maintaining their maximal speed. During the 100 m race, athletes keep accelerating the wheelchair since they reach their maximal speed at the end of the race [9]. It would then be interesting to develop two models, one with a fixed shoulder, which would be useful for long-distance races, and another one with a moving shoulder, for the short-distance races.

In this context, the desired optimal position of the athlete in his wheelchair aims at: reducing the aerodynamic drag, maximizing the mechanical work exerted on the pushing rim at each pushing cycle, and minimizing the loss of energy especially for long races. Thus, this study focused on obtaining the optimal position of the athlete on the wheelchair for two scenarios, using dynamics models. The first model aimed at maximizing the wheelchair speed during the permanent regime, for long-distance races. The second one focused on minimizing the time to reach 100 m.

#### 2 Material and methods

Two models have been designed, with different objectives and hypotheses. The main difference between the two models was the motion of the shoulder, which was fixed for the long-distance race model but not for the short distance model. This was due to the fact that athletes had a higher torso amplitude, therefore, a high shoulder motion, during the first accelerations, than when they reached their cruising speed [9].

#### 2.1 Long distance race model

This model aimed at finding the best position of the athlete in its wheelchair to reach a maximal permanent speed.

For that objective, the energy provided by the athlete to the wheelchair was computed, depending on the position of the athlete on the wheelchair and its velocity. The optimal position was defined when this energy was maximized.

This long-distance race model (Fig. 1 on the left) was implemented with the following assumptions:

- The athlete's trunk was fixed.
- The shoulder and elbow joints were assimilated as pivots.
- The hand was assimilated as a point on the pushing rim.
- The model only considered the rear wheels and the athlete.
- The speed during the pushing cycle was considered constant.

The model varied the position of the athlete's shoulder relatively to the centre of the wheel. For each position, the total energy of the pushing cycle needed for a specific speed of the wheelchair was calculated using an incremental method. A pushing cycle was defined as the elementary cycle of action of the athlete on the pushing rim, between the first contact with the pushing rim until the athlete removed his hands. The rotation was split into 360 points, each point corresponding to one degree (angle  $\alpha$ ). Between two points on the pushing rim, the point representing the wrist made a circular rotation of one degree.

The rolling resistance was modelled using the model developed by Masson [10] and the aerodynamic drag was adapted from the Samozino's model [11].

For each point on the pushing rim, the position of each joint centre and the angles of the joints were computed. Then, the joint speed was also computed using the information of a second point on the pushing rim close to the first one. Since, we did not have access to data about the forces generated by the athletes at their shoulder and torque during a propulsion, the values have been based on the literature. It was then possible to obtain the shoulder [12] and elbow torques [13,14] based on the joint speeds. The torque value used in our study was the maximal torque exerted. The torques developed at the joints by the muscles created a force on the pushing rim, and the tangential part of this pushing force was considered.

The scalar product of the tangential force and the motion of one degree of the hand on the pushing rim corresponded to the energy supplied by the athlete to the wheelchair between two points. The energy computed for all the points of the push cycle was summed up to find the total energy. If the total energy (W) was greater than the energy required to counteract the resistive forces ( $E_n$ ), the algorithm increased the wheelchair speed and restarted the calculation for the same position, but with the new wheelchair speed. Otherwise, the maximum speed was reached.

#### 2.2 Sprint race model

The long-distance model used the maximal speed to obtain the best position. However, for a 100 m race, the maximal speed was reached only on the last few meters. Indeed, since the athletes keep accelerating all along the 100 m, the maximal speed reachable by the athlete in the wheelchair might not be the best criterion for performance. Consequently, another model was developed to determine the best position in the wheelchair in order to minimize the time needed to reach 100 m.

When starting, athletes tended to be mobile with their trunk whereas their trunk was fixed when they reached their cruise speed [9]. It was therefore considered in this second model since the angle between the trunk and the wheelchair influenced the shoulder and elbow angles.

This sprint race model (Fig. 1 on the right) was implemented with the following assumptions:

- The shoulder and elbow joints were assimilated as pivots.
- The speed of the wheelchair was assumed to be constant during one pushing cycle.
- The athlete considered is in the category T53 which meant that he did not have control over his abdominal muscles to maintain his posture, and that his back was bound to the wheelchair.

The same methodology than the first model, in terms of computation of the tangential force, was applied to this second one with the difference that the shoulder position was not fixed. Also, the parameter computed to stop the iterating process was the distance reached by the athlete. The model stopped when the distance travelled (Length) reached 100 m. The time needed to reach these 100 m was then computed.

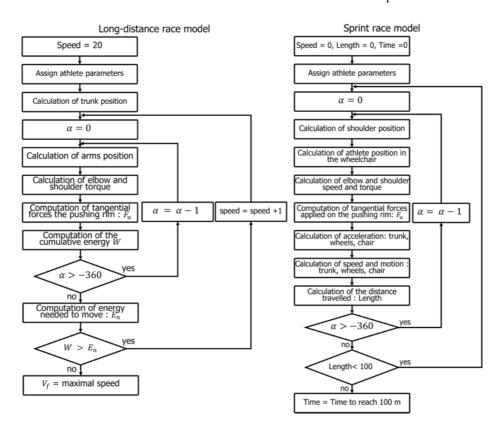
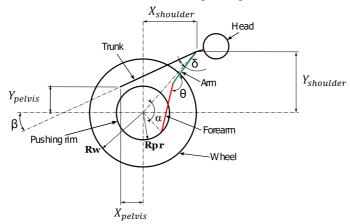


Fig. 1. Workflow of the long-distance race model (left) and sprint race model (right)

#### 2.3 Dataset considered to test the models

To test the different models, a dataset including the entry parameters of the models was defined. First, the anthropometric data such as torso, arm, and forearm length were needed. As an example, to try out the models, test-data were taken from one of the authors. The arm length was set at 250 mm, the forearm length at 300 mm and the torso length at 550 mm. The parameters of the wheelchair needed were only the rear wheel diameter and the pushing rim diameter. The rear wheel diameter was set at 700 mm and the pushing rim diameter at 350 mm, which were based on the observation of an existing racing wheelchair.

Then, constraints had to be implemented. First, the degrees of freedom of the joints were set between a minimal and a maximal value: the torso angle was set between  $\beta_{min}=20^\circ$  and  $\beta_{max}=45^\circ$ , the shoulder joint angle between  $\delta_{min}=-30^\circ$  and  $\delta_{max}=170^\circ$ , and the elbow joint angle between  $\theta_{min}=20^\circ$  and  $\theta_{max}=175^\circ$ . These values were also used as an example to test the model. These angles (Fig. 2) were used to determine when the hand is in contact with the pushing rim.



**Fig. 2.** Definition of the angles used in the model.  $\alpha$  is the angle between the hand and the shoulder,  $\beta$  is the torso angle,  $\delta$  the shoulder joint angle, and  $\theta$  the elbow joint angle.  $X_{shoulder}$ ,  $Y_{shoulder}$ ,  $X_{pelvis}$ , and  $X_{pelvis}$  are respectively the horizontal and vertical position of the shoulder and the pelvis, with respect to the wheel centre.

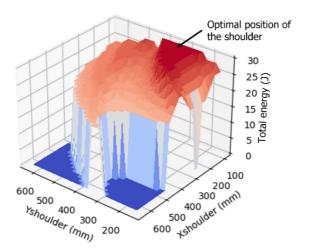
Then, for each position of the hand on the pushing rim, the position of the other joints was computed as well as their instantaneous speed. The torques generated by the joints were then taken from the literature based on their angle and speed [12–14]. The torque considered was assumed to be the maximal torque obtained in those studies. This assumption could be discussed as the subjects of these studies were able-bodied whereas our study concerned para-athletes.

#### 3 Results

#### 3.1 Results of the long-distance race model

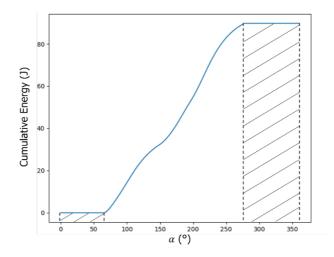
Given the dataset considered, the long-distance race model was first tested. The first iteration reached a maximal speed of 8.33 m/s with a position of the shoulder relative to the centre of the wheel of  $X_{shoulder} = 100mm$  and  $Y_{shoulder} = 350mm$  using a step of  $\Delta = 20mm$  in both axes. Then, with a second optimisation using a step of  $\Delta = 2mm$ , the optimal position of the shoulder relative to the wheel centre was  $X_{shoulder} = 80mm$ , and  $Y_{shoulder} = 332mm$ . With this position, the maximal velocity reached by the athlete on his wheelchair was 8.88 m/s. This first result seemed to be in accordance with the actual position used by the athletes.

These iterations allowed us to map the optimal position of the shoulder of the athlete as a function of the total energy generated by athlete on his wheelchair (Fig. 3). It was observed that, for the athlete with the anthropometry and the wheelchair geometry used in this study, the optimal position of his shoulder would be between  $X_{shoulder} = 0$  to 90mm, and  $Y_{shoulder} = 300$ to 400mm.



**Fig. 3.** Map of the total energy supplied by the athlete to the wheelchair as a function of the horizontal and vertical position of the shoulder.

The evolution of the total energy as a function of the  $\alpha$  angle is presented on Fig. 4. It can be observed that there was no evolution of the total energy between  $\alpha = 0^{\circ}$  to 70° and between  $\alpha = 260^{\circ}$  to 360°. This was because the athlete was not able to hold on the pushing rim for these angles. This incapacity to grab the pushing rim was linked to the anthropometry of the athlete and his position on the wheelchair.



**Fig. 4.** Evolution of the simulated total energy as a function of the angle  $\alpha$ . The hatched area represents the angles for which the athlete was not able to grab the pushing rim.

#### 3.2 Results of the sprint race model

Using the same dataset, the sprint race model was then tested. The first iteration of the second model used the following position of the pelvis, with respect to the wheel centre:  $X_{pelvis} = 100$  mm and  $Y_{pelvis} = 50$  mm. The time needed to reach 100 m was then of  $t_{100m} = 21.21s$  and the maximal wheelchair velocity reached was  $v_{max} = 5.8 \, m/s$ .

After the optimisation of the position of the pelvis on the wheelchair, the time returned was 14.35s to complete the race. The maximal velocity of the wheelchair reached was then 9.08 m/s. The optimal position of the hip was  $X_{pelvis} = 386mm$  and  $Y_{pelvis} = 386mm$ 

The results of the first model suggested that the shoulder should be between  $Y_{shoulder} = 300$  and 400 mm to reach an optimal position. The second model returned about the same vertical position of the pelvis. This means that the athlete should have different settings depending on the race he's participating in.

The wheelchair speed as a function of the time was computed (Fig. 5) and it can be observed that it followed an exponential curve. Also, the speed always increased, and it can be suspected that there would be a plateau when the maximal speed was reached but this maximal speed was not reached during the 100 m race.

#### 4 Discussion

The values of maximal wheelchair speed obtained with the Long-distance race model was  $v_{max} = 8.88$  m/s. This value was close to the one found in the study by Sauret et al. [9] where experimental data have been captured using IMU. Indeed, they found a maximal speed of 8.86 m/s for the first athlete and 8.09 m/s for the second athlete. However, the position of the shoulder used by these athletes was more towards the front

of the chair rather than close to the wheel centre, which was suggested by our optimization

Concerning sprint races, in the literature [9], two athletes performed a 100 m race and the best of them had a time of 15.38s and reached his maximal speed of 8.86 m/s at the end of the race. Since it was a short race, the speed of the athlete kept increasing, until the 100 m. However, for longer races, a plateau could be seen. This phenomenon was also observed in our results with the sprint race model (Fig. 5) where the speed kept increasing during the first hundred meters.

Therefore, the use of both models can be useful to coaches and athletes, as it may give the optimal positions of the pelvis and the shoulder to obtain either the best acceleration pattern and/or the highest maximal speed during both short and long-distance races. Nevertheless, the comfort of the athlete should also be considered when selecting the optimal position on the wheelchair. Indeed, if the athlete would not be able to maintain this optimal position due do discomfort, then this position would not be optimal for the performance of the athlete.

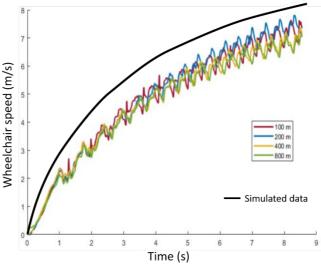
The dataset used in this study was not taken from a Paralympic athlete, which could explain the differences found in this study. Indeed, the parameters used in the models were based on the measurements made on one of the authors. Differences could exist between the actual measurements of a Paralympic athlete and the model could gain in accuracy with a dataset coming from an actual athlete.

One of the hypotheses made for the first model was that the shoulder was not moving during the propulsion when the athlete reached his maximal speed. This was the case for the T53 athletes since they do not have control over their abdominal muscles but T54 athletes do have control over those muscles. Therefore, they had a non-negligible motion of their shoulders with an amplitude of almost 40° [9]. The optimal position of the shoulder found with the first model could also be used for the athletes with control over their abdominal muscles as a mean position of the shoulder during propulsion, and the athletes would be able to generate even more mechanical work during the cycle.

The speed of the wheelchair was deemed constant throughout the pushing cycle; however, as it can be observed on the Fig. 5, the hand was not in contact with the pushing rim during the whole cycle. Neglecting this allowed us to compute the optimal acceleration curve of the wheelchair. For coaches, this might be seen as the optimal mean acceleration curve that the athletes should aim to achieve.

As it can be observed on the Fig. 5, the sprint race model over-estimated the speed of actual athletes. Those differences could be due to the fact that the athletes may not have the optimal position in their wheelchairs, or the wheelchair might be poorly designed. It might also be due to the different hypotheses made in this study. Indeed, the torques computed in this study were based on the literature, and they were computed based on the angles and rotational speeds of the joints. The torque applied was then the maximal torque the joint could supply. Even if the athletes were the best in their field, they were not able to provide maximal torque during the whole cycle. Also, only the in-plane components of the force applied by the hand on the pushing rim was considered. However, the force applied by the hand is three-dimensional [15]. Therefore, the out-of-plane part of the force was neglected, and this force may be detrimental to the propulsion, so neglecting it may result in an increased wheelchair speed. This could be

one of the improvements to be made in the future. Considering the three components of the force on the pushing rim dynamics should result in a diminution of the wheelchair speed, and the simulated data would probably match better the experimental ones.



**Fig. 5.** Superposition of our simulated data (black) with the sprint race model and the experimental data of the literature [9] (with authorization) for different distances.

The model would also be more realistic if the wheelchair speed was not supposed constant during the pushing cycle. Indeed, considering the freewheel phase of the pushing cycle would lead to a more realistic propulsion phase. Also, going from a 2D model to a 3D model could help considering more than the tangential component of the force of the hand on the pushing rim [15]. Another possible evolution of the models would be to consider the shoulder and elbow as actual joints with multiple degrees of freedom [16], which would result in a more complex, but more realistic model of the propulsion.

Finally, after the optimization proposed, a last model was developed to determine the optimal position of the centre of mass of the athlete and his wheelchair set in order to avoid the wheelie effect during the acceleration phase [17].

#### 5 Conclusion

The models developed in this study allowed to compute the optimal position of both the shoulder and the pelvis of the athlete to generate the maximal mechanical work on the pushing rim, leading to a maximal wheelchair speed. This could help athletes, professionals as beginners, to set up and/or optimize their wheelchair and to be seated at the optimal position that would allow the maximal speed. The second model might also be helpful for coaches, who could set the optimal acceleration curve as an objective to their athletes.

To improve the models, the 3D force applied by the hand on the pushing rim might be considered. Also, considering the freewheel phase of the pushing cycle could provide information about a more realistic pushing cycle.

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#### Annex

The formulas used to compute the kinematic parameters of the model are presented in this annex. This allowed us to find the shoulders and elbow angles.

The first step to calculate the kinematics parameters of the Long-distance model was based on the current position of the shoulder centre ( $X_{shoulder}$  and  $Y_{shoulder}$  respectively the horizontal and vertical position):

$$inc = \arctan\left(\frac{Y_{shoulder}}{X_{shoulder}}\right); D = \sqrt{(X_{shoulder}^2 + Y_{shoulder}^2)}$$

For the sprint model, the first step to compute the kinematic parameters was based on the dimension of the torso  $(L_{torso})$ , the horizontal and vertical position of the pelvis (respectively  $X_{pelvis}$  and  $Y_{pelvis}$ ), and  $\beta$ , the angle between the torso and a horizontal line (figure attached).

$$X_{shoulder} = L_{torso} \cos(\beta) - X_{pelvis}; Y_{shoulder} = L_{torso} \sin(\beta) + Y_{pelvis}$$
  
 $inc = \arctan\left(\frac{Y_{shoulder}}{X_{shoulder}}\right); D = \sqrt{(X_{shoulder}^2 + Y_{shoulder}^2)}$ 

Then, the next step was the same for both models. The parameters used were:  $\alpha$ , the

$$v = \alpha + \frac{\pi}{2} - inc$$

$$L = \sqrt{(R_{mc}^2 + D^2 - 2DR_{mc}\cos(\nu))}$$

$$\theta = -\arccos\left(\frac{L_{arm}^2 + L_{forearm}^2 - L^2}{2L_{arm}L_{forearm}}\right)$$

$$\psi = -\arctan\left(\frac{R_{mc}\sin(\nu)}{D - R_{mc}\cos(\nu)}\right)$$

$$\mu = -\arccos\left(\frac{L^2 + L_{arm}^2 - L_{forearm}^2}{2L_{arm}L}\right) + \psi$$

$$\varphi = \mu + \theta$$

$$\delta = inc + \mu - \beta$$

